Timed Petri Nets and BQOs (for CS 735- Formal Models for Concurrent and Asynchronous Systems)

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Meet Taraviya
Indian Institute of Technology Bombay

Timed Petri Nets and BQOs

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CS 735, Spring 2017-18
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Outline

Recap

1. Places P
2. Transtions T
3. $T \times P \equiv Pre : T \rightarrow 2^P$
4. $P \times T \equiv Post : T \rightarrow 2^P$
5. Marking $M : P \rightarrow \mathbb{N}$
6. Firing Condition $\forall p \in Pre(t), M(p) > 0$

Example

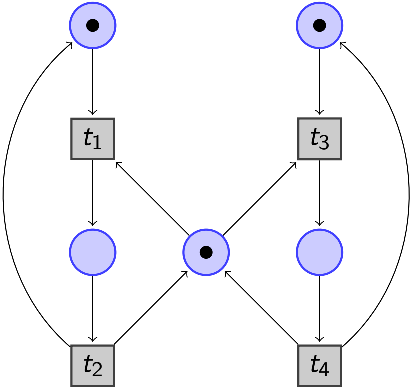

Figure 1: Mutual Exclusion

Initial Marking $M_0 = (a, b, d)$

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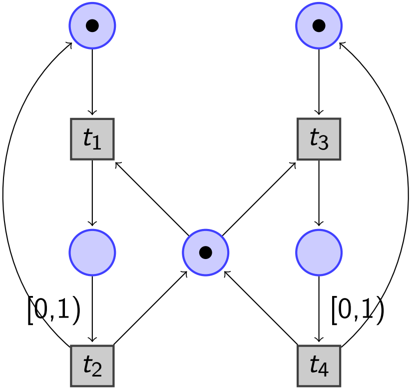

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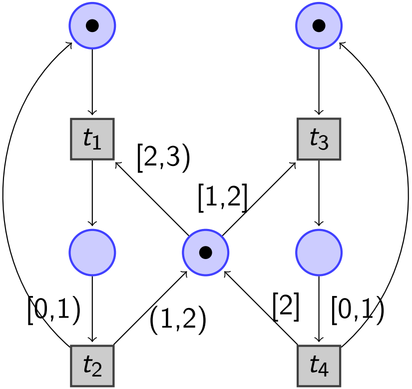

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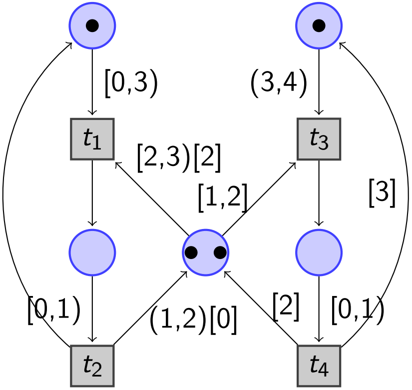

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Timed Petri Net

1. Add time to tokens

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2. Label each arc by intervals, Multiple intervals possible

Formal Definition

A timed Petri net (TPN) $\mathcal{N} = P, T, Pre, Post, \lambda)$ where:

- P is a finite set of places,
- T is a finite set of transitions with $P \cap T = \phi$
- $Pre, Post : T \times P \rightarrow (\mathcal{I}^{\oplus})$ [1] where $\mathcal{I}$ is set of intervals (closed integral bounds, right-unbounded)
- $\lambda : T \rightarrow \Sigma \cup \{\epsilon\}$ is a labelling function

[1]The operation $A^{\oplus}$ is called Bag. $Bag : A \rightarrow \mathbb{N}$

Semantics

Marking

Marking $M : P \to (\mathbb{R}^{\oplus}_{\geq 0})$

Alternatively we can also write it as $M \in (P \times \mathbb{R}_{\geq 0})^{\oplus 2}$

We say $M \leq M' \iff \forall q \in P \times \mathbb{R}_{\geq 0}, M(q) \leq M'(q)$

[2] We will abuse the notations indicating M((p,x))=M(p)(x)

Transitions

Delay Transitions

For $\delta \in \mathbb{R}_{\geq 0}$ and $M = ((p_1, x_1), (p_2, x_2), \ldots, (p_i, x_i))$
$Mz \xrightarrow{\delta} M' \iff M' =$
$((p_1, x_1 + \delta), (p_2, x_2 + \delta), \ldots, (p_i, x_i + \delta))$

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Discrete Transition

For $t \in T$, $M \xrightarrow{\lambda(t)} M'$

[3]The time of the new token after the transition is fired is selected non-deterministically

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  - B_2 denotes the new tokens after transition. $B_2 \models Post(t)$[3]

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  - B_2 denotes the new tokens after transition. $B_2 \models Post(t)$[3]
- Thus, $M' = M - B_1 + B_2$

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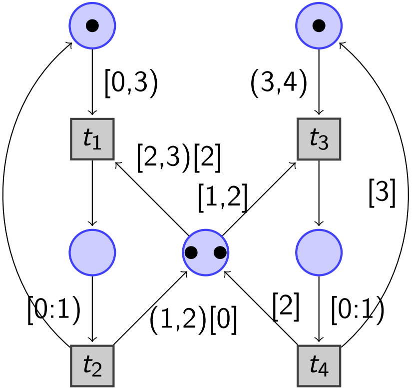

Figure 2: Mutual Exclusion

Marking $M_0 = ((a,0),(b,0),(d,0),(d,0))$
$M_0 \xrightarrow{2} ((a,2),(b,2),(d,2),(d,2)) \xrightarrow{t_1} ((b,2),(c,0))$

Semantics (contd...)

Firing Sequence

Thus it will be $(t_1, \tau_1), (t_2, \tau_2) \ldots$
The transition sequence which ouccurs is:
$M_{in} \xrightarrow{\tau_1} M_1 \xrightarrow{t_1} M_2 \xrightarrow{\tau_2 - \tau_1} M_2 \xrightarrow{t_2} \ldots$

Coverability

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- **Existential zones**

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- $\bar{P} : m^+ \rightarrow P$ called a *placing*, which maps each token to a place
- $D : m^* \times m^* \rightarrow \mathbb{N} \cup \{\infty\}$ called a *difference bound matrix*, defines restriction on the ages of the tokens

[1] $n^+ = \{1, 2, ..., n\}$
[2] $n^* = \{0, 1, ..., n\}$

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- *M satisfies Z*, written $M \models Z$, if $M, h \models Z$ for some h.
- $[\![Z]\!] = \{M; M \models Z\}$
- $[\![Z]\!]$ is **upward closed**

Existential zones

$$\left(2, \bar{P} = (p_1, p_2), \begin{array}{c|ccc} - & 0 & 1 & 2 \\ \hline 0 & - & -2 & -3 \\ 1 & 4 & - & 0 \\ 2 & 5 & 2 & - \end{array}\right)$$ represents all markings

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- $x_2 - x_1 \leq 2$

Lemma 1

For an existential zone Z and a marking M, it is decidable whether $M \models Z$

Normal and consistent Existential Zones

- An existential zone Z is said to be *normal* if for each $i, j, k \in m^*$, we have $D(j,i) \leq D(j,k) + D(k,i)$.
- An existential zone Z is said to be *consistent* if $[\![Z]\!] \neq \Phi$.

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- Is $\preceq$ a *well* quasi order?
- Yes it is!
- To prove this we prove that it is a **better quasi order** (bqo).

Better quasi orders

Barrier

- $\beta \subset \mathbb{N}^{<\omega}$ is called a barrier if

Examples

- $\{(a,b) | b > a\}$
- $\{(a,b,c) | c > b > a\}$
- $\{(a,b) | b > a > 1\} \cup \{1\}$

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[3] $\mathbb{N}^{\omega}$ is the set of all *infinite* strictly increasing sequences over $\mathbb{N}$
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  - There are s_1, s_2 such that $tail(s_1) \ll s_2$ and $f(s_1) \preceq f(s_2)$

[1] $tail(s_1)$: sequence after deleting first element of s_1

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Definition Better quasi orders

- $(A, \preceq)$ is a better quasi order if every A-pattern is good.

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[1] $X \sqsubseteq Y \iff \forall x \in X \exists y \in Y : x \preceq y$

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  - But $(\mathcal{P}(X), \sqsubseteq)$ is not a wqo
  - Infinite antichain: $X_i = \{(i, j)|j > i\}$

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- Tokens in the same bag have the same fractional part
- Tokens in B_0 have fractional part zero
- Fractional part in B_{i+1} > Fractional part in B_i
- B_{n+1} contains tokens with age larger than m

[1] m is the largest constant appearing in the intervals

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- Existential regions $(\mathcal{R} = ((P \times N)^B)^*, (=^B)^*)$ is bqo

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Computing Pre_{δ}

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  - $D'(0, i) = 0$
  - $D'(j, i) = D(j, i)$ if $j \neq 0$

Computing Predecessors

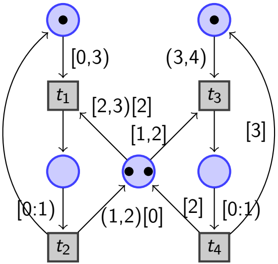

Figure 3: Mutual Exclusion

- Example $Z =$

$4, \bar{P}$

—	0	1	2	3	4
0	—	−1	−1	−1	−1
1	1	—	0	0	0
2	1	0	—	0	0
3	1	0	0	—	0
4	1	0	0	0	—

↓

$4, \bar{P}$

—	0	1	2	3	4
0	—	**0**	**0**	**0**	**0**
1	1	—	0	0	0
2	1	0	—	0	0
3	1	0	0	—	0
4	1	0	0	0	—

Computing Predecessors

Computing Pre_t

- Conjunction $Z \otimes (\mathcal{I}, i)$

Computing Predecessors

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- Conjunction $Z \otimes (\mathcal{I}, i)$
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—	0	1	2	3	4
0	—	0	0	0	0
1	2	—	0	0	0
2	2	0	—	0	0
3	2	0	0	—	0
4	2	0	0	0	—

$\otimes ([1:3], 1) \rightarrow$

—	0	1	2	3	4
0	—	−1	0	0	0
1	2	—	0	0	0
2	2	0	—	0	0
3	2	0	0	—	0
4	2	0	0	0	—

Computing Predecessors

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Computing Predecessors

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Computing Predecessors

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$$\left(5, (A, B, D, D, A), \begin{array}{c|cccccc} - & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & - & 0 & 0 & 0 & 0 & -1 \\ 1 & 2 & - & 0 & 0 & 0 & \infty \\ 2 & 2 & 0 & - & 0 & 0 & \infty \\ 3 & 2 & 0 & 0 & - & 0 & \infty \\ 4 & 2 & 0 & 0 & 0 & - & \infty \\ 5 & 2 & \infty & \infty & \infty & \infty & - \end{array} \right)$$

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- Abstraction $Z \setminus i$

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- Abstraction $Z \setminus i$
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$$\left(3, (A, B, D), \begin{array}{c|cccc} - & 0 & 1 & 2 & 3 \\ \hline 0 & - & 4 & 3 & 1 \\ 1 & 4 & - & 0 & 0 \\ 2 & 3 & 0 & - & 0 \\ 3 & 1 & 0 & 0 & - \end{array} \right)$$

Computing Predecessors

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  - $Z \otimes (\mathcal{J}_{h(i_1}, i_1) \otimes (\mathcal{J}_{h(i_2}, i_2)... \otimes (\mathcal{J}_{h(i_n}, i_n)$ is consistent
  - $Z_1 = Z \setminus i_1 \setminus i_2 ... \setminus i_n$
  - $Z' = Z_1 \oplus (p_1, \mathcal{I}_1) \oplus (p_2, \mathcal{I}_2) ... \oplus (p_k, \mathcal{I}_k)$

Coverability and Termination

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- What about **termination?**
- For termination, we need effective *Post* computability
- *Post* computability can be proved similarly
- Hence termination is **decidable**

Fast Growing Hierarchy

Recap

Gregorczyk Hierarchy: $f_{k+1}(n) = {f_k}^n(n)$
$f_\omega(n) = f_n(n)$[4]

Fast Growing Hierarchy

We construct the hierarchy as follows:

- $\omega + \omega = \omega.2$, Similarly,$\omega + \omega.(n-1) = \omega.n$
- $\omega.\omega = \omega^2$. Applying finite times: $\omega.\omega^{n-1} = \omega^n$
- $\omega^\omega = \omega.\omega.\omega. \ldots$
- $\omega^{\omega^{\omega^{\cdots}}}$ and so on

[4] ω is like infinity(smallest supremum over natural numbers)

Length Function Theorem

Let g be a smooth control function eventually bounded by a function in $\mathbb{F}_\gamma$, and let A be an exponential nwqo with maximal order type $< \omega^{\beta+1}$. Then $L_{(A,g)}$ is bounded by a function in

- $\mathbb{F}_\beta$ if $\gamma < \omega$ (e.g. if g is primitive-recursive) and $\beta \geq \omega$,
- $\mathbb{F}_{\gamma+\beta}$ if $\gamma \geq 2, \beta < \omega$

Maximal Order Type

Definition

Longest linearization of a bad sequence isomorphic to an ordering

- $o(\Gamma_k) = k$
- $o(\Gamma_0^*) = \omega^0$
- $o(\Gamma_{k+1}^*) = \omega^{\omega^k}$
- $o(A \oplus B) = o(A) \oplus o(B)$
- $o(A \otimes B) = o(A) \otimes o(B)$

Computing Complexity: $L_{A,g}$

- $o(P) = |p|$
- $o(N_{\leq m}) = m \implies o(P \times [m]) = |p|m$
- $o(Bag(P \times [m])) = \omega * (|p|m)$
- $o((Bag(P \times [m]))^*) = \omega^{\omega^{\omega * (|p|m)}}$
- $o(\cup(Bag(P \times [m]))^*) = \omega^{\omega^{\omega * (|p|m)}}.\omega$
- Thus $o(\mathcal{Z}) = \omega^{\omega^{\omega^{|p|m}}}$

Computing Complexity: Predecessor

- Calculation of Predecessor: For time lapse move, the Pre calculation is just updating the markings which can be done in $\mathbb{F}_1$
- For discrete transition: Even for that, the steps required will be the number of tokens which is still in $\mathbb{F}_1$

Final Complexity

- We see that the complexity of the algorithm is thus dependent mainly on the maximum length of the bad sequence.
- By Length-Function Theorem, we get the complexity is $\mathbb{F}_{\omega^{\omega^{|p|m}}}$
- Generalising we get, the complexity as $\mathbb{F}_{\omega^{\omega^{\omega}}}$.

Conclusion

- Existential zones of timed petri Nets form BQO
- BQOs are WQOs thus timed petri nets are WSTS over existential zones as the states and transition as that of the timed petri nets
- Coverability and Termination are Decidable
- Complexity of the Coverability and Termination algorithms is $\mathbb{F}_{\omega^{\omega^{\omega}}}$